TECHNICAL RESEARCH REPORT

Variability ordering for the backlog in buffer models fed by on-off fluid sources

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		Form Approved OMB No. 0704-0188

Report Documentation Page

1. REPORT DATE **2003**	2. REPORT TYPE	3. DATES COVERED **00-00-2003 to 00-00-2003**
4. TITLE AND SUBTITLE **Variability ordering for the backlog in buffer models fed by on-off fluid sources**		5a. CONTRACT NUMBER
		5b. GRANT NUMBER
		5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)		5d. PROJECT NUMBER
		5e. TASK NUMBER
		5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **University of Maryland,Electrical Engineering Department,Institute for Systems Research,College Park,MD,20742**		8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		10. SPONSOR/MONITOR'S ACRONYM(S)
		11. SPONSOR/MONITOR'S REPORT NUMBER(S)
12. DISTRIBUTION/AVAILABILITY STATEMENT **Approved for public release; distribution unlimited**		
13. SUPPLEMENTARY NOTES		
14. ABSTRACT **see report**		
15. SUBJECT TERMS		

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES **17**	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**			

Variability ordering for the backlog
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Abstract

In the context buffer models fed by independent on-off fluid sources, we explore conditions under which "determinism minimizes the stationary backlog." These comparison results are couched in terms of the convex ordering for distributions. We show that increased variability in the on-duration rv results in greater variability of the corresponding backlog. While it appears that in general increased variability in the off-duration rv does not necessarily imply greater variability of the backlog, it is however the case when the on-period duration rv B is exponentially distributed. The discussion is organized around a representation of the stationary backlog in terms of the stationary waiting time rv for an auxiliary stable $GI|GI|1$ queue.

1 Introduction

Consider the following popular model for evaluating the performance of ATM multiplexers: An independent on-off fluid source with peak rate r is offered to an infinite capacity buffer which is drained at the constant rate c. The statistics of such an on-off fluid source are fully determined by a pair of independent $\mathbb{R}_+$-valued random variables (rvs) B and I describing the generic on-period and off-period durations, respectively. If $rp < c < r$ where p is

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the asymptotic fraction of time that the source is active, then there exists a non-identically zero $\mathbb{R}_+$-valued rv V, known as the stationary backlog, which can be used to characterize the buffer ocuppancy level in steady state.

As for classical queueing systems, it is of some interest to understand how (the distribution of) the backlog $V(B, I)$ varies with (that of) B and I (where we write $V(B, I)$ to acknowledge the fact that the statistics of V are determined by the distributions of the rvs B and I). In this paper we are specifically concerned with the following *external* monotonicity properties [11]: For $k = 1, 2$, let $V(B^{(k)}, I^{(k)})$ denote the stationary backlog induced by the on-off fluid source $(B^{(k)}, I^{(k)})$. We seek conditions on the comparability of the rvs $B^{(1)}$ and $B^{(2)}$, on one hand, and of the rvs $I^{(1)}$ and $I^{(2)}$, on the other hand, which would ensure the comparability of the backlog rvs $V(B^{(1)}, I^{(1)})$ and $V(B^{(2)}, I^{(2)})$. In particular, we wish to determine when the comparisons $B^{(1)} \leq_b B^{(2)}$ and $I^{(1)} \leq_i I^{(2)}$ for some stochastic orderings $\leq_b$ and $\leq_i$ imply a similar comparison, say $V(B^{(1)}, I^{(1)}) \leq_v V(B^{(2)}, I^{(2)})$, possibly for some other stochastic ordering $\leq_v$.

Work along these lines has been reported in the literature for a wide range of queueing systems and in varying degree of completeness. The most comprehensive set of results was perhaps obtained for the queues $GI|GI|1$ and $GI|GI|c$ (with c finite) [11, 13]. These monotonicity results are given in both transient and steady–state regimes for several performance measures, including queue size, customer waiting time and workload [9, 11, 13]. They are typically expressed in terms of one of the "standard" integral orders on the set of probability distributions: For $\mathbb{R}$–valued random variables X and Y, we say that X is smaller than Y in the strong stochastic (resp. convex, increasing convex) ordering if

$$\mathbf{E}\left[\varphi(X)\right] \leq \mathbf{E}\left[\varphi(Y)\right] \tag{1}$$

for all mappings $\varphi : \mathbb{R} \to \mathbb{R}$ which are monotone increasing (resp. convex, increasing and convex) provided the expectations in (1) exist. In that case we write $X \leq_{st} Y$ (resp. $X \leq_{cx} Y$, $X \leq_{icx} Y$). Additional material on these orderings can be found in the monographs [9, 10, 11, 12].

Here we discuss a number of results under the comparability conditions $B_1 \leq_{cx} B_2$ and $I_1 \leq_{cx} I_2$. The use of the convex ordering $\leq_{cx}$ (rather than $\leq_{st}$ or $\leq_{icx}$) in these conditions can be traced to a representation of the stationary backlog in terms of the stationary waiting time rv for an auxiliary stable $GI|GI|1$ queue; this representation is developed in Section 4. The

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sought–after comparison results would constitute analogs for buffer models fed by on-off fluid sources of the fact that "determinism minimizes waiting time" in $GI|GI|1$ queues [3] (and references therein).

In Section 5 we present a first (and easy) comparison result to the effect that $V(B^{(1)}, I^{(1)}) \leq_{icx} V(B^{(2)}, I^{(2)})$ whenever $B^{(1)} \leq_{cx} B^{(2)}$ with $I^{(1)} = I^{(2)} \equiv I$. In other words, increased variability in the on-duration rv results in greater variability of the corresponding backlog. This result (given in Proposition 5.1) is a simple consequence of the above-mentionned respresentation result and of standard external monotonicity results for $GI|GI|1$ queues (summarized in Section 3).

Next, in Section 6 we consider the case when

$$I^{(1)} \leq_{cx} I^{(2)} \quad \text{with} \quad B^{(1)} = B^{(2)} \equiv B \tag{2}$$

and show why the approach underlying Proposition 5.1 is inadequate. In Sections 7 and 8 we try to remedy this difficulty by imposing additional conditions to (2). While we are able to establish the desired result under this augmented set of conditions, they are shown to be too strong in that together with (2), they imply $I^{(1)} =_{st} I^{(2)}$.

Nevertheless, we are still able to make progress in the case when the on-period duration rv B is exponentially distributed. To do so, in Section 9 we take advantage of well-known results for the $GI|M|1$ queue and derive an *explicit* expression for the distribution function of the stationary backlog. This expression has already appeared elsewhere [4, 5] for a model equivalent to the one considered here, and is then used in Section 10 to establish the comparison $V(B, I^{(1)}) \leq_{st} V(B, I^{(2)})$ under (2) for an exponentially distributed on-period rv B. As of this writing it is still an open problem as to whether a comparison result holds under (2) in its general form.

A word on the notation used in this paper: We find it convenient to define all the rvs of interest on some common probability triple $(\Omega, \mathcal{F}, \mathbf{P})$. Two $\mathbb{R}$–valued rvs X and Y are said to be equal in law if they have the same distribution, a fact we denote by $X =_{st} Y$. For any $\alpha > 0$, we denote by E_α any rv which is exponentially distributed rv with parameter α. For any integrable $\mathbb{R}_+$–valued rv X, the forward recurrence time $X^\star$ is defined as the rv with integrated tail distribution given by

$$\mathbf{P}[X^\star > x] := \frac{1}{\mathbf{E}[X]} \int_x^\infty \mathbf{P}[X > t]\, dt, \quad x \geq 0. \tag{3}$$

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We recall that $X^\star$ has a finite first moment if $\mathbf{E}\left[X^2\right] < \infty$, with

$$\mathbf{E}\left[X^\star\right] = \frac{\mathbf{E}\left[X^2\right]}{2\mathbf{E}\left[X\right]}.$$

2 On-off sources

An on-off source of peak rate r is described by a succession of cycles, each such cycle comprising an off-period followed by an on-period. During the on-periods the source is active and produces fluid at constant rate r; the source is silent during the off-periods: For each $n = 0, 1, \ldots$, let B_n and I_n denote the durations of the on-period and off-period in the $(n+1)^{st}$ cycle, respectively. Thus, if the sequence of epochs $\{T_n, \ n = 0, 1, \ldots\}$ denote the beginning of successive cycles, with $T_0 := 0$ we have $T_{n+1} := \sum_{\ell=0}^n I_\ell + B_\ell$ for each $n = 0, 1, \ldots$. The activity of the source is then described by the $\{0, 1\}$-valued process $\{\xi(t), \ t \geq 0\}$ given by

$$\xi(t) := \sum_{n=0}^\infty \mathbf{1}\left[T_n + I_n \leq t < T_{n+1}\right], \quad t \geq 0, \tag{4}$$

with the source active (resp. silent) at time t if $\xi(t) = 1$ (resp. $\xi(t) = 0$).

An *independent on-off* source is one for which (i) the $\mathbb{R}_+$-valued rvs $\{I_n, \ n = 1, \ldots\}$ and $\{B_n, \ n = 1, \ldots\}$ are mutually independent rvs which are independent of the pair of rvs I_0 and B_0 associated with the initial cycle; and (ii) the rvs $\{I_n, \ n = 1, \ldots\}$ (resp. $\{B_n, \ n = 1, \ldots\}$) are i.i.d. rvs with generic off-period duration rv I (resp. on-period duration rv B). Throughout the generic rvs B and I are assumed to be independent $\mathbb{R}_+$-valued rvs such that $0 < \mathbf{E}\left[B\right], \mathbf{E}\left[I\right] < \infty$, and we simply refer to the independent on-off process just defined as the on-off source (B, I).

In general, the activity process (4) is not stationary unless the rvs I_0 and B_0 are selected appropriately. We do so by using the following variation on constructions given in [1, 8]: With

$$p := \frac{\mathbf{E}\left[B\right]}{\mathbf{E}\left[B\right] + \mathbf{E}\left[I\right]}, \tag{5}$$

we introduce the $\{0, 1\}$-valued rv U given by

$$\mathbf{P}\left[U = 1\right] = p = 1 - \mathbf{P}\left[U = 0\right]. \tag{6}$$

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A stationary version of (4), still denoted $\{\xi(t),\ t \geq 0\}$, is now obtained by selecting (I_0, B_0) so that

$$(I_0, B_0) =_{st} (0, B^\star)U + (I^\star, B)(1 - U) \tag{7}$$

with rvs U, B, $B^\star$ and $I^\star$ taken to be mutually independent and independent of the rvs $\{B_n, I_n,\ n = 1, \ldots\}$. Here, the rvs $B^\star$ and $I^\star$ are the forward recurrence times associated with B and I, respectively, as defined by (3).

3 Variability ordering in $GI|GI|1$ queues

In this section we summarize some useful facts and notation concerning $GI|GI|1$ queues. Consider a standard $GI|GI|1$ queue with generic service time σ and interarrival time τ; these rvs are assumed integrable. For each $n = 0, 1, \ldots$, let W_n denote the waiting time (in buffer) of the nth customer. Under the stability condition

$$\mathbf{E}\left[\sigma\right] < \mathbf{E}\left[\tau\right], \tag{8}$$

there exists an $\mathbb{R}_+$–valued rv W such that $W_n \Longrightarrow_n W$ irrespective of W_0. We refer to the rv W as the *stationary waiting time* rv. It can be characterized as the supremum of a random walk with i.i.d. increments [2, Prop. 1.1, p. 181], namely

$$W =_{st} \left(\sup_{n=0,1,\ldots} \sum_{\ell=1}^{n} (\sigma_\ell - \tau_{\ell+1}) \right)^+ \tag{9}$$

where $\{\sigma, \sigma_\ell,\ \ell = 0, 1, \ldots\}$ and $\{\tau, \tau_{\ell+1},\ \ell = 0, 1, \ldots\}$ are mutually independent sequences of i.i.d. rvs (where $x^+ = \max(x, 0)$ for any scalar x). We also recall [2, Thm. 2.1, p. 184] [6] that under the assumed condition $\mathbf{E}\left[\tau\right] < \infty$, it holds that

$$\mathbf{E}\left[W\right] < \infty \quad \text{if and only if} \quad \mathbf{E}\left[\sigma^2\right] < \infty. \tag{10}$$

In the sequel we write $W(\sigma, \tau)$ for the stationary waiting time rv (9) associated with the standard $GI|GI|1$ queue with generic service time σ and interarrival time τ. We also introduce the stationary delay rv $D(\sigma, \tau)$ given by

$$D(\sigma, \tau) =_{st} W(\sigma, \tau) + \sigma \tag{11}$$

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where the rvs $W(\sigma, \tau)$ and σ are taken to be independent. In what follows we shall make use of the fact that the rv $W(\sigma, \tau)$ is a solution to the distributional equation

$$W =_{st} (W + \sigma - \tau)^+ \tag{12}$$

with W, σ and τ mutually independent rvs.

For $GI|GI|1$ queues it is well known [3, 9, 11] that "determinism minimizes waiting times," a fact which can be formalized by Theorem 5.2.1 of [11, p. 80] (when combined with (10)).

Proposition 3.1 *Consider two stable $GI|GI|1$ queues with integrable generic service time $\sigma^{(k)}$ and interarrival time $\tau^{(k)}$ (thus $\mathbf{E}\left[\sigma^{(k)}\right] < \mathbf{E}\left[\tau^{(k)}\right]$) so that $\mathbf{E}\left[|\sigma^{(k)}|^2\right] < \infty$ $(k = 1, 2)$. If*

$$\sigma^{(1)} \leq_{cx} \sigma^{(2)} \quad \text{and} \quad \tau^{(1)} \leq_{cx} \tau^{(2)}, \tag{13}$$

then it holds that

$$W(\sigma^{(1)}, \tau^{(1)}) \leq_{icx} W(\sigma^{(2)}, \tau^{(2)}). \tag{14}$$

Under condition (13), $\mathbf{E}\left[\sigma^{(1)}\right] = \mathbf{E}\left[\sigma^{(2)}\right]$ and $\mathbf{E}\left[\tau^{(1)}\right] = \mathbf{E}\left[\tau^{(2)}\right]$, and the two $GI|GI|1$ queues have the same stability condition.

4 The stationary backlog

Consider the stationary version $\{\xi(t), \ t \geq 0\}$ of the on-off source (B, I) with peak rate r as described in Section 2. The total amount $A(t)$ of fluid generated in $[0, t)$ by this on-off source is given by

$$A(t) = r \int_0^t \xi(s)ds, \quad t \geq 0. \tag{15}$$

If we offer this on-off source $\{A(t), \ t \geq 0\}$ to an infinite capacity buffer drained at the constant rate c, then under the condition $c < r$, a backlog results in the amount $V(t)$ at time $t \geq 0$. With a system initially empty, standard arguments show that

$$V(t) = \sup_{0 \leq s \leq t} \left(A(t) - A(s) - c(t - s) \right). \tag{16}$$

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Under the stability condition

$$rp < c \qquad (17)$$

with p given by (5), there exists a non-identically zero $\mathbb{R}_+$-valued rv V such that $V(t) \Longrightarrow_t V$ irrespective of the initial backlog $V(0)$. It is known that

$$V =_{st} \sup_{t \geq 0} (A(t) - ct) \qquad (18)$$

with $\{A(t), \ t \geq 0\}$ given by (15). We refer to the rv V as the *stationary backlog*.

The stationary backlog (18) can be related to the stationary waiting time of an auxiliary $GI|GI|1$ queue. To that end, consider the $\mathbb{R}$-valued rvs $\{X_\ell, \ \ell = 0, 1, \ldots\}$ defined by

$$X_\ell := (r - c)B_\ell - cI_\ell, \quad \ell = 0, 1, \ldots \qquad (19)$$

and set

$$M := \left(\sup_{n=1,2,\ldots} \sum_{\ell=1}^{n} X_\ell \right)^+ \qquad (20)$$

where the rvs $\{I_n, B_n, \ n = 0, 1, \ldots\}$ are as specified in the construction of the stationary version of $\{\xi(t), \ t \geq 0\}$ in Section 2. Under the enforced assumptions, the rv X_0 is independent of the i.i.d. rvs $\{X_\ell, \ \ell = 1, 2, \ldots\}$. While $X_\ell =_{st} (r - c)B - cI$ for all $\ell = 1, 2, \ldots$, it follows from (7) that

$$X_0 =_{st} (r - c)B^\star U + ((r - c)B - cI^\star)(1 - U). \qquad (21)$$

Proposition 4.1 *With the rv X_0 taken independent of the rv M, we have*

$$V =_{st} (X_0 + M)^+ . \qquad (22)$$

Proof. Apply Proposition 4.1 in [1, p. 17] with mapping $h : \mathbb{R}_+ \to \mathbb{R}$ taken to be $h = 0$. This choice for h is simultaneously superadditive and subadditive, and satisfies both conditions (H1)-(H2). Thus, both Claims 1 and 2 together yield

$$V =_{st} \left(\sup_{n=0,1,\ldots} \sum_{\ell=0}^{n} X_\ell \right)^+ . \qquad (23)$$

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It is plain that

$$V =_{st} \left(X_0 + \max\left(0, \sup_{n=1,2,\dots} \sum_{\ell=1}^{n} X_\ell \right) \right)^+ \tag{24}$$

and the desired conclusion (22) immediately follows. ∎

Upon comparing (9) and (20), we observe the equivalence

$$M =_{st} W((r - c)B, cI) \tag{25}$$

where in the notation developed in Section 3, $W((r - c)B, cI)$ denotes the stationary waiting time for the $GI|GI|1$ queue with generic service time $(r - c)B$ and interarrival service time cI. Note that (17) is equivalent to (8) with the identification $\sigma = (r - c)B$ and $\tau = cI$.

Reporting (25) into the representation (22) and making use of (21), we find

$$\begin{aligned} V \ =_{st} \ & ((r - c)B^\star + W((r - c)B, cI))\, U \\ & + ((r - c)B - cI^\star + W((r - c)B, cI))^+ (1 - U). \end{aligned} \tag{26}$$

Sometimes we shall find it useful to write $V(B, I)$, $M(B, I)$ and $X_0(B, I)$ for V, M and X_0, respectively, to indicate the dependence of these quantities on the rvs B and I.

5   A first comparison result

Whenever we discuss a comparison result, we shall assume the following framework: Consider the buffer model with drain rate c which is fed by the on-off fluid sources $(B^{(k)}, I^{(k)})$ with peak rate r $(k = 1, 2)$. We assume

$$p_k r < c \quad \text{and} \quad \mathbf{E}\left[|B^{(k)}|^2 \right] < \infty, \quad k = 1, 2 \tag{27}$$

with p_k given by (5) for the on-off source $(B^{(k)}, I^{(k)})$. The following result indicates in what sense "determinism in the on-period duration minimizes the stationary backlog."

Proposition 5.1 *If*

$$I^{(1)} = I_{(2)} \equiv I \quad and \quad B^{(1)} \leq_{cx} B_{(2)}, \tag{28}$$

then

$$V(B^{(1)}, I) \leq_{icx} V(B^{(2)}, I). \tag{29}$$

Proof. In view of (25), we can invoke Proposition 3.1 to conclude under (28) that

$$M(B^{(1)}, I) \leq_{icx} M(B^{(2)}, I). \tag{30}$$

On the other hand, it is a simple matter to check that (28) implies $B^{(1)\star} \leq_{st} B^{(2)\star}$, hence

$$B^{(1)\star} \leq_{icx} B^{(2)\star}. \tag{31}$$

Under the enforced assumptions, we have $p_1 = p_2$ (whence $U^{(1)} =_{st} U^{(2)}$), and with the help of (21), we see from (28) and (31) that

$$X_0(B^{(1)}, I) \leq_{icx} X_0(B^{(2)}, I). \tag{32}$$

The fact that the convex increasing ordering is closed under convolution yields

$$M(B^{(1)}, I) + X_0(B^{(1)}, I) \leq_{icx} M(B^{(2)}, I) + X_0(B^{(2)}, I) \tag{33}$$

and the desired conclusion (29) is now a simple consequence of (22) and of the fact that convex increasing transformations preserve the convex increasing ordering. ∎

6 A difficulty

We now turn to the situation when $B^{(1)} = B^{(2)} \equiv B$. In view of Proposition 3.1, it is natural to seek a comparison result under the assumption

$$I^{(1)} \leq_{cx} I^{(2)} \tag{34}$$

with the hope that a result similar to (29) will materialize in the form

$$V(B, I^{(1)}) \leq_{icx} V(B, I^{(2)}). \tag{35}$$

To explore the validity of (35), we proceed as in the proof of Proposition 5.1: Under the enforced assumptions (34), Proposition 3.1 yields $M(B, I^{(1)}) \leq_{icx} M(B, I^{(2)})$ and the equality $p_1 = p_2$ still holds and again we have $U^{(1)} =_{st} U^{(2)}$.[1]

With the help of (21), we see that the analog of (32) will indeed hold in the form

$$X_0(B, I^{(1)}) \leq_{icx} X_0(B, I^{(2)}) \tag{36}$$

provided we can show that

$$-I^{(1)\star} \leq_{icx} -I^{(2)\star}. \tag{37}$$

Once this comparison established, the proof of (35) can then be completed in a rather routine manner along the lines of the proof of Proposition 5.1.

However, now (34) implies $I^{(1)\star} \leq_{st} I^{(2)\star}$, whence $-I^{(2)\star} \leq_{st} -I^{(1)\star}$ and the comparison

$$-I^{(2)\star} \leq_{icx} -I^{(1)\star} \tag{38}$$

follows. The validity of both (37) and (38) yields $-I^{(2)\star} =_{st} -I^{(1)\star}$, or equivalently, $I^{(1)\star} =_{st} I^{(2)\star}$, whence the equality $I^{(1)} =_{st} I^{(2)}$. In other words, the proof of (35) under (34) with *distinct* $I^{(1)}$ and $I^{(2)}$ cannot pass through (36), and is not as straightforward as was the proof of Proposition 5.1. We devote the next two sections to trying to establish the comparison under additional conditions.

7 A dead-end

In order to circumvent the difficulty discussed in the last section, it might be tempting to argue as follows: With the distributional equation (12) in mind, the form of (26) suggests the possibility that (35) will hold under (34) provided the conditions

$$-I^{(1)\star} \leq_{icx} -I^{(1)} \tag{39}$$

and

$$-I^{(2)} \leq_{icx} -I^{(2)\star} \tag{40}$$

are added.

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[1]In Sections 7 and 8 we denote any such $\{0, 1\}$-valued rv by U.

Indeed, (12) and (39) together lead to

$$\left((r-c)B - cI^{(1)\star} + W((r-c)B, cI^{(1)})\right)^+$$
$$\leq_{icx} \left((r-c)B - cI^{(1)} + W((r-c)B, cI^{(1)})\right)^+$$
$$= W((r-c)B, cI^{(1)}), \tag{41}$$

so that

$$V(B, I^{(1)}) \leq_{icx} (r-c)B^\star U + W((r-c)B, cI^{(1)}) \tag{42}$$

upon making use of (26). Similarly, (12) and (40) combine to give

$$\left((r-c)B - cI^{(2)\star} + W((r-c)B, cI^{(2)})\right)^+$$
$$\geq_{icx} \left((r-c)B - cI^{(2)} + W((r-c)B, cI^{(2)})\right)^+$$
$$= W((r-c)B, cI^{(2)}), \tag{43}$$

and the representation (26) now yields

$$V(B, I^{(2)}) \geq_{icx} (r-c)B^\star U + W((r-c)B, cI^{(2)}). \tag{44}$$

Combining (42) and (44) readily implies (35) once we note by Proposition 3.1, that $W((r-c)B, cI^{(1)}) \leq_{icx} W((r-c)B, cI^{(2)})$ holds under condition (34).

Unfortunately, conditions (39) and (40) together with (34) again force the equality $I^{(1)} =_{st} I^{(2)}$: From the discussion in Section 6 we already know that (34) implies the comparison $-I^{(2)\star} \leq_{icx} -I^{(1)\star}$. On the other hand (34) is equivalent to $-I^{(1)} \leq_{cx} -I^{(2)}$, whence $-I^{(1)\star} \leq_{icx} -I^{(2)\star}$ as a result of (39) and (40). Thus, $-I^{(1)\star} =_{st} -I^{(2)\star}$, i.e., $I^{(1)\star} =_{st} I^{(2)\star}$ and the equality $I^{(1)} =_{st} I^{(2)}$ follows since $\mathbf{E}\left[I^{(1)}\right] = \mathbf{E}\left[I^{(2)}\right]$ under (34).

8 Yet another dead-end

The discussion in Sections 6 and 7 suggests that the comparison (35) under (34) may not be valid unless additional assumptions are made on B, such conditions possibly involving $I^{(1)}$ and $I^{(2)}$. We explore this point further in the next two sections.

As we again return to (12), we add to (34) the conditions

$$(r-c)B - cI^{(1)\star} \leq_{icx} 0 \tag{45}$$

and

$$0 \leq_{icx} (r-c)B - cI^{(2)\star}. \tag{46}$$

Then,

$$\left((r-c)B - cI^{(1)\star} + W((r-c)B, cI^{(1)})\right)^+$$
$$\leq_{icx} \quad W((r-c)B, cI^{(1)}) \tag{47}$$

while

$$\left((r-c)B - cI^{(2)} + W((r-c)B, cI^{(2)})\right)^+$$
$$\geq_{icx} \quad W((r-c)B, cI^{(2)}). \tag{48}$$

Making use of the representation (26) now yields

$$V(B, I^{(1)}) \quad \leq_{icx} \quad (r-c)B^\star U + W((r-c)B, cI^{(1)}) \tag{49}$$

and

$$V(B, I^{(2)}) \quad \geq_{icx} \quad (r-c)B^\star U + W((r-c)B, cI^{(2)}). \tag{50}$$

Combining (49) and (50) readily implies (35) since $W((r-c)B, cI^{(1)}) \leq_{icx} W((r-c)B, cI^{(2)})$ by virtue of Proposition 3.1 under condition (34).

But conditions (45) and (46) together imply

$$c\mathbf{E}\left[I^{(2)\star}\right] \leq (r-c)\mathbf{E}\left[B\right] \leq c\mathbf{E}\left[I^{(1)\star}\right]. \tag{51}$$

On the other hand, (34) was shown to yield (38) in Section 6, whence

$$\mathbf{E}\left[I^{(1)\star}\right] \leq \mathbf{E}\left[I^{(2)\star}\right]. \tag{52}$$

Thus, $\mathbf{E}\left[I^{(1)\star}\right] = \mathbf{E}\left[I^{(2)\star}\right]$ while at the same time we have $I^{(1)\star} \leq_{st} I^{(2)\star}$. Consequently, $I^{(1)\star} =_{st} I^{(2)\star}$ and the equality $I^{(1)} =_{st} I^{(2)}$ again follows!

9 Exponential on-periods

To make progress we consider the situation when the on-period durations are exponentially distributed: The buffer model with drain rate c is now fed by an on-off source (B, I) with peak rate r where $B =_{st} E_\beta$ for some parameter β, so that

$$(r - c)B =_{st} E_\mu \quad \text{with} \quad \mu = (r - c)^{-1}\beta. \tag{53}$$

Note that $B^\star =_{st} B$ and that (22) yields

$$V =_{st} \left(\tilde{X}_0 + (r - c)B + M \right)^+ \tag{54}$$

with rvs $\tilde{X}_0$, B and M mutually independent, and $\tilde{X}_0$ characterized by

$$\tilde{X}_0 =_{st} -cI^\star(1 - U). \tag{55}$$

The key observation is that M is now the equilibrium waiting time in a stable $GI|M|1$ queue with generic service time $(r - c)B =_{st} E_\mu$ and interarrival time cI. Thus, $M + (r - c)B$ can now be interpreted as the stationary delay $D((r - c)B, cI)$ in that $GI|M|1$ queue. It is well known that the stationary delay rv in a stable $GI|M|1$ queue is exponentially distributed [7]: More precisely,

$$D((r - c)B, cI) =_{st} E_{\xi(1-\mu)} \tag{56}$$

where ξ the unique solution of the non-linear equation

$$\xi = \mathbf{E}\left[e^{-\mu(1-\xi)cI} \right], \quad 0 < \xi < 1. \tag{57}$$

Consequently, (54) can now be rewritten as

$$V =_{st} \left(\tilde{X}_0 + E_{\xi(1-\mu)} \right)^+ \tag{58}$$

with independent rvs $\tilde{X}_0$ and $E_{\xi(1-\mu)}$. We are now in position to evaluate the distribution function of the backlog when activity periods are exponentially distributed. The following result was obtained in [4, 5] in a somewhat different form for a model equivalent to the one considered here:

Proposition 9.1 *Consider the buffer model with drain rate c when fed by an on-off source (B, I) with peak rate r. Assume that $B =_{st} E_\beta$ for some parameter $\beta > 0$. Then, with μ given by (53), it holds that*

$$\mathbf{P}[V > t] = \frac{pr}{c} e^{-\mu(1-\xi)t}, \quad t \geq 0. \tag{59}$$

Proof. Fix $t \geq 0$. We shall show below that

$$\mathbf{P}\left[V > t\right] = \left(p + \frac{(1-p)}{\mu c \mathbf{E}\left[I\right]}\right) e^{-\mu(1-\xi)t}. \tag{60}$$

The expression (59) is now a straightforward consequence of this last relation as we recall the expression (5) for p and the fact that $\beta = \mathbf{E}\left[B\right]^{-1}$ in (53).

Now turning to the derivation of (60), we use (55) with (58) to get

$$
\begin{aligned}
\mathbf{P}\left[V > t\right] &= p\mathbf{P}\left[E_{\xi(1-\mu)} > t\right] + (1-p)\mathbf{P}\left[E_{\xi(1-\mu)} - cI^\star > t\right] \\
&= pe^{-\mu(1-\xi)t} + (1-p)\mathbf{P}\left[E_{\xi(1-\mu)} > cI^\star + t\right].
\end{aligned} \tag{61}
$$

By the independence of $E_{\xi(1-\mu)}$ and $I^\star$, we see that

$$
\begin{aligned}
\mathbf{P}\left[E_{\xi(1-\mu)} > cI^\star + t\right] &= \mathbf{E}\left[e^{-\mu(1-\xi)(cI^\star + t)}\right] \\
&= e^{-\mu(1-\xi)t}\mathbf{E}\left[e^{-\mu(1-\xi)cI^\star}\right]
\end{aligned} \tag{62}
$$

with

$$\mathbf{E}\left[e^{-\mu(1-\xi)cI^\star}\right] = \frac{1 - \mathbf{E}\left[e^{-\mu(1-\xi)cI}\right]}{\mu(1-\xi)c\mathbf{E}\left[I\right]} = \frac{1}{\mu c \mathbf{E}\left[I\right]} \tag{63}$$

where the first equality is the standard formula that relates the Laplace–Stieltjes transform of $I^\star$ to that of I, and the second equality made use of (57). Reporting (63) into (62) we readily get (60) via (61). ∎

10 Comparing with exponential on-periods

The main comparison result is contained in the following:

Proposition 10.1 *With $B =_{st} E_\beta$ for some parameter β, if*

$$B^{(1)} = B^{(2)} \equiv B \quad and \quad I^{(1)} \leq_{cx} I^{(2)}, \tag{64}$$

then

$$V(B, I^{(1)}) \leq_{st} V(B, I^{(2)}). \tag{65}$$

Proof. Under the enforced assumptions, we have $\mu_1 = \mu_2 \equiv \mu$ and $p_1 = p_2 \equiv p$. Thus, by Proposition 9.1 we get the desired conclusion (65) provided $\mathbf{P}\left[V(B, I^{(1)}) > t\right] \leq \mathbf{P}\left[V(B, I^{(2)}) > t\right]$ for all $t \geq 0$, i.e.,

$$\frac{pr}{c} e^{-\mu(1-\xi_1)t} \leq \frac{pr}{c} e^{-\mu(1-\xi_2)t}, \quad t \geq 0 \tag{66}$$

where for each $i = 1, 2$, ξ_i is the unique solution of the non-linear equation

$$\xi = \mathbf{E}\left[e^{-\mu(1-\xi)cI_i}\right], \quad 0 < \xi < 1. \tag{67}$$

The requirement (66) is obviously equivalent to

$$\xi_1 \leq \xi_2. \tag{68}$$

The validity of (68) is a simple consequence of the defining relations (67) for ξ_1 and ξ_2, and of condition (64) through the inequality

$$\mathbf{E}\left[e^{-\mu(1-\xi)cI_1}\right] \leq \mathbf{E}\left[e^{-\mu(1-\xi)cI_2}\right], \quad 0 < \xi < 1. \tag{69}$$

∎

11   Acknowledgement

References

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